"Does biology care about bits?"
From Mattingly and Kamino et al Nat Phys *2021 and 2026.*

What is the relationship between velocity and signal?
Signal is defined as

$$s(t) = \frac{d}{dt}\log(c).$$

In a shallow gradient,

$$c(x) = c_0(1 + gx).$$

where g is the steepness. Thus,

$$s(t) = gv_x(t),$$

where v_x is the velocity up the concentration gradient.

How did they calculate information rate from signal to kinase activity $\dot{I}_{s\to a}$?
The information rate between Gaussian processes related by a linear map is

$$\dot{I}(X,Y) = \frac{1}{4\pi}\int \log\left(1 + \frac{S(\omega)|K(\omega)|^2}{N(\omega)}\right)d\omega,$$

where S, K, and N are the power spectra of the input signal, the linear response function, and the noise in the output. Let's understand this expression by (i) showing that the mutual information is related to the Pearson correlation coefficient for two Gaussian variables related by a linear map, and (ii) making an analogy between the variance and the power spectrum.

Consider the system $y = kx + \xi$, where the variables x, y, and the noise ξ are all Gaussian. In other words, x and y follow a bivariate normal distribution. From inferring position errors in drosophila embryogenesis, the entropy of a Gaussian variable is

$$H(x) = -\langle -\frac{1}{2}\log(2\pi\sigma_x^2) - \frac{(x-\mu_x)^2}{2\sigma_x^2}\rangle = \frac{1}{2}\log(2\pi e\sigma_x^2).$$

The joint probability distribution involves the covariance matrix, and the variance in the normalization factor gets replaced by the determinant of the covariance matrix. Since the diagonal terms of the covariance matrix is proportional to the Pearson correlation coefficient ρ, the determinant involves a term $-\rho^2$,

$$H(x,y) = \frac{1}{2}\log((2\pi e)^2|\Sigma|) = 1 + \log(2\pi) + \log\big(\sigma_x\sigma_y\big) + \frac{1}{2}\log(1-\rho^2).$$

Combining the entropies,

$$I(x,y) = H(x) + H(y) - H(x,y) = -\frac{1}{2}\log(1-\rho^2).$$

This expression can be written in terms of variances as follows. The Pearson correlation coefficient is just a normalized covariance, which in this case is

$$\sigma_{xy} = \langle x(kx+\xi)\rangle = \langle kx^2\rangle = k\sigma_x^2.$$

Writing out the correlation coefficient,

$$\rho^2 = \frac{\sigma_{xy}^2}{\sigma_x^2\sigma_y^2} = \frac{k^2\sigma_x^4}{\sigma_x^2(k^2\sigma_x^2+\sigma_n^2)} = \frac{k^2\sigma_x^2}{k^2\sigma_x^2+\sigma_n^2}.$$

Hence, the mutual information between x and y is

$$I(x,y) = \frac{1}{2}\log\left(1 + \frac{k^2\sigma_x^2}{\sigma_n^2}\right).$$

For white noise, the power spectrum is flat and equal to the variance at all frequencies. We can extend this relationship to an analogy, $k \to K(\omega)$, $\sigma_x^2 \to S(\omega)$, and $\sigma_n^2 \to N(\omega)$. The analogy can be made exact using Fourier transforms and complex analysis. Thus, the information rate can be interpreted as the ratio of the signal-induced power versus noise power, integrated over frequencies.